Degree-bounded factorizations of bipartite multigraphs and of pseudographs

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March 22, 2007

Abstract

For $d \geq 1$, $s \geq 0$ a $(d, d+s)$-*graph* is a graph whose degrees all lie in the interval $\{d, d+1, \ldots, d+s\}$. For $r \geq 1$, $a \geq 0$ an $(r, r+1)$-*factor* of a graph G is a spanning $(r, r+a)$-subgraph of G. An $(r, r+a)$-*factorization* of a graph G is a decomposition of G into edge-disjoint $(r, r+a)$-factors.

We prove a number of results about $(r, r+a)$-factorizations of $(d, d+s)$-bipartite multigraphs and of $(d, d+s)$-pseudographs (multigraphs with loops permitted). For example, for $t \geq 1$ let $\beta(r, s, a, t)$ be the least integer such that, if $d \geq \beta(r, s, a, t)$ then every $(d, d+s)$-bipartite multigraph G has an $(r, r+a)$-factorization into x $(r, r+a)$-factors for at least t different values of x. Then we show that

$$\beta(r, s, a, t) = r \left\lceil \frac{tr+s-1}{a} \right\rceil + (t-1)r.$$

Similarly, for $t \geq 1$ let $\psi(r, s, a, t)$ be the least integer such that if $d \geq \psi(r, s, a, t)$ then each $(d, d+s)$-pseudograph has an $(r, r+a)$-factorization into x $(r, r+a)$-factors for at least t different values of x. We show that, if r and a are even, then $\psi(r, s, a, t)$ is given by the same formula.

We use this to give tight bounds for $\psi(r, s, a, t)$ when r and a are not both even. Finally, we consider the corresponding functions for multigraphs without loops, and for simple graphs.

1 Introduction

For $d \geq 1$, $s \geq 0$ a $(d, d+s)$-*graph* is a graph whose degrees all lie in the interval $\{d, d+1, \ldots, d+s\}$. For $r \geq 1$, $a \geq 0$ an $(r, r+a)$-*factor* of a graph G is a spanning $(r, r+a)$-subgraph of G. An $(r, r+a)$-*factorization* of a graph G is a decomposition of G into edge-disjoint $(r, r+a)$-factors. An $(r, r+a)$-factorization is also described less precisely as a *degree-bounded factorization* of G.

A survey paper dealing with degree-bounded factorizations was published by Akiyama and Kano in 1985 [1], and recent surveys by Plummer [14, 15] also deal with degree-bounded factorizations. Further important papers are by Akiyama and Kano [2], Kano [10] and Cai [3]. For some recent work by the present author, see [6], [7] and [8].

Bipartite multigraphs are the simplest kind of graph to consider for some factorization problems; in particular, we are able without much difficulty to obtain exact results for the questions about degree-bounded factorizations we consider here. *Pseudographs* are multigraphs in which loops are permitted; a loop counts two towards the degree of the vertex it is on. There is a well-known connection between Eulerian pseudographs and bipartite multigraphs. We exploit this connection to deduce some exact and some approximate results about the analogous questions concerning certain kinds of degree-bounded factorizations of pseudographs. Finally we draw attention to the various implications for similar questions about simple graphs and about multigraphs (without loops).

In Section 2 we discuss bipartite multigraphs. In Section 3 we apply the results from Section 2 to pseudographs; direct application of the bipartite multigraph results leads to good results about $(r,r+a)$-factorizations of $(d,d+s)$-pseudographs in the case when r and $r+a$ are both even. In Section 4 we extend these results to the cases when r and $r+a$ are not both even. In Section 5 we examine the implication of these results for the analogous problems about multigraphs without loops and about simple graphs.

Before concluding our introduction, let us draw attention to the following lemma about $(r,r+a)$-factorizations of $(d,d+s)$-pseudographs.

Lemma 1. *Let r be a positive integer and s and a be non-negative integers. Let G be a $(d,d+s)$-pseudograph with at least one vertex of degree d and at least one vertex of degree $d+s$. Suppose that G has an $(r,r+a)$-factorization with exactly x $(r,r+a)$-factors. Then*

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}.$$

Proof. Let v be a vertex of degree $d+s$. Then $x(r+a) \geq \mathrm{d}(v) = d+s$, so $x \geq \frac{d+s}{r+a}$. Similarly, if w is a vertex of degree d, then $xr \leq \mathrm{d}(w) = d$, so $x \leq \frac{d}{r}$. □

2 Factorizing bipartite multigraphs

In our first theorem we show that, given $d,r,a,s,$ there is a large interval $I = I(d,r,a,s) = \left[\frac{d}{r+a}, \frac{d+s}{r}\right]$ which has the property that there exist $(d,d+s)$-bipartite multigraphs G which have $(r,r+a)$-factorizations into x $(r,r+a)$-factors if and only if $x \in I$, and a smaller interval $J = J(d,r,a,s) = \left[\frac{d+s}{r+a}, \frac{d}{r}\right]$ which has the property that all $(d,d+s)$-bipartite multigraphs have an $(r,r+a)$-factorization into x $(r,r+a)$-factors if and only if $x \in J$. Similar but more specialized results for simple graphs were proved in [6] and [8].

An invaluable tool in our proofs is the following easy result due to McDiarmid [12] and de Werra [16, 17, 18]. For a graph G, an *edge-colouring* of G is a map

$\phi : E(G) \mapsto \mathcal{C}$, where $\mathcal{C}$ is a set of colours. An edge-colouring ϕ of G is *equitable* if

$$| \, |\alpha(v)| - |\beta(v)| \, | \leq 1$$

for each vertex $v \in V(G)$ and pair $\alpha, \beta \in \mathcal{C}$, where $\alpha(v)$ and $\beta(v)$ are the sets of edges incident with v coloured α and β respectively (a loop on v counts two towards $|\alpha(v)|$ or $|\beta(v)|$, respectively).

The result of McDiarmid and de Werra is:

Lemma 2. *Let k be a positive integer and let G be a bipartite multigraph. Then G has an equitable edge-colouring with k colours.*

Our first theorem is:

Theorem 3. *Let d, r and x be positive integers, and let a, s be non-negative integers.*

(i) If

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$$

then every $(d, d+s)$-bipartite multigraph has an $(r, r+a)$-factorization into x $(r, r+a)$-factors.

(ii) If

$$x \in \left[\frac{d}{r+a}, \frac{d+s}{r+a}\right) \cup \left(\frac{d}{r}, \frac{d+s}{r}\right]$$

then some $(d, d+s)$-bipartite multigraphs do and some do not have an $(r, r+a)$-factorization into x $(r, r+a)$-factors.

(iii) If

$$x \notin \left[\frac{d}{r+a}, \frac{d+s}{r}\right]$$

then no $(d, d+s)$-bipartite multigraph has an $(r, r+a)$-factorization into x $(r, r+a)$-factors.

Proof. (i) Suppose that $\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$. Then $r \leq \frac{d}{x} \leq \frac{d+s}{x} \leq r+a$. Let G be a $(d, d+s)$-bipartite multigraph. By Lemma 2, G has an equitable edge-colouring with x colours. Since $r \leq \frac{d}{x} \leq \frac{d+s}{x} \leq r+a$, it follows that each colour class is an $(r, r+a)$-factor of G. Thus G has an $(r, r+a)$-factorization into x $(r, r+a)$-factors.

(ii) Let $x \in \left[\frac{d}{r+a}, \frac{d+s}{r+a}\right) \cup \left(\frac{d}{r}, \frac{d+s}{r}\right]$.

First we show that there are $(d, d+s)$-bipartite multigraphs which do have an $(r, r+a)$-factorization into x $(r, r+a)$-factors. Since $\frac{d}{r+a} \leq x \leq \frac{d+s}{r}$, if

$a \geq 1$ then there are integers a_1 and s_1 with $0 \leq a_1 \leq a$ and $0 \leq s_1 \leq s$ such that

$$\frac{d}{r+a} \leq \frac{d+s_1}{r+a_1+1} \leq x \leq \frac{d+s_1}{r+a_1} \leq \frac{d+s}{r}$$

so that $x(r+a_1) \leq d+s_1 \leq x(r+a_1+1)$. Therefore, there are integers $x_1 \geq 0$, $x_2 \geq 0$ and $x_1+x_2 = x$ such that $x_1(r+a_1)+x_2(r+a_1+1) = d+s$, or, putting $a_1+1 = a_2$,

$$x_1(r+a_1)+x_2(r+a_2) = d+s_1$$

with $0 \leq a_1 \leq a_2 \leq a$. This equation also holds if $a = 0$ for some s_1 with $0 \leq s_1 \leq s$, for then $x = d+s_1$, so we can have $a_1 = a_2 = 0$, $x_1 = x$, $x_2 = 0$. Let $F_1, \ldots, F_{x_1}$ be $(r+a_1)$-regular bipartite multigraphs with the same bipartition (V_1, V_2) of their vertex sets, and let $F_{x_1+1}, \ldots, F_{x_1+x_2}$ be $(r+a_2)$-regular bipartite multigraphs also with the bipartition (V_1, V_2). Then let $G = \bigcup_{i=1}^{x} F_i$. Then G is regular of degree $(r+a_1)x_1+(r+a_2)x_2 = d+s_1$. Thus G is a $(d, d+s)$-bipartite multigraph which has an $(r, r+a)$-factorization into x $(r, r+a)$-factors.

Next we show that there are $(d, d+s)$-bipartite multigraphs which do not have an $(r, r+a)$-factorization into x $(r, r+a)$-factors.

Firstly, let $x \in \left[\frac{d}{r+a}, \frac{d+s}{r+a}\right)$ and let G be a $(d+s)$-regular bipartite multigraph. The average degree over all the factors of the vertices of G in a decomposition of G into x factors is $\frac{d+s}{x}$. But $\dfrac{d+s}{x} > \dfrac{d+s}{(d+s)/(r+a)} = r+a$, so the factors cannot all be $(r, r+a)$-factors.

Secondly, let $x \in \left(\frac{d}{r}, \frac{d+s}{r}\right]$ and let G be a d-regular bipartite multigraph. The average degree over all the factors of the vertices of G in a decomposition of G into x factors is $\frac{d}{x}$. But $\dfrac{d}{x} < \dfrac{d}{(d/r)} = r$, so the factors cannot all be $(r, r+a)$-factors.

(iii) If $x < \frac{d}{r+a}$ then $x(r+a) < d$. Thus the union of x $(r, r+a)$-bipartite multigraphs has maximum degree less than d, and so no $(d, d+s)$-bipartite multigraph has a decomposition into x $(r, r+a)$-factors. Similarly, if $x > \frac{d+s}{r}$, then $xr > d+s$. Thus the union of x $(r, r+a)$-bipartite multigraphs has minimum degree greater than $d+s$, so no $(d, d+s)$-bipartite multigraph has a decomposition into x $(r, r+a)$-factors.

$\square$

We note the following two corollaries of Theorem 3.

Corollary 4. *Let d, r, x be positive integers and let s be a non-negative integer. Then every $(d, d+s)$-bipartite multigraph has an $(r, r+a)$-factorization into x $(r, r+a)$-factors if and only if*

$$x \in \left[\frac{d+s}{r+a}, \frac{d}{a}\right].$$

Corollary 5. *Let d, r, x be positive integers and let s be a non-negative integer. Then there is some $(d, d+s)$-bipartite multigraph which has an $(r, r+a)$-factorization into x $(r, r+a)$-factors if and only if*

$$x \in \left[\frac{d}{r+a}, \frac{d+s}{r}\right].$$

Next we apply Theorem 3. For positive integers r, a, t and non-negative integer s, let $\beta(r,s,a,t)$ be the smallest integer such that, for each integer $d \geq \beta(r,s,a,t)$, each $(d, d+s)$-bipartite multigraph has an $(r, r+a)$-factorization with x $(r, r+a)$-factors for at least t different values of x. In Theorem 6 we evaluate $\beta(r,s,a,t)$.

Theorem 6. *Let integers r, a, t be positive and s be non-negative. Then*

$$\beta(r,s,a,t) = \frac{r}{a}(tr+s+c) + (t-1)r,$$

where c is such that $a \mid tr+s+c$ and $-1 \leq c \leq a-2$.

Theorem 11 below about pseudographs seems to read exactly the same, but note that there c is even and we have $0 \leq \frac{c}{2} \leq \frac{a}{2} - 1$.

In [8] an exact result for simple graphs when $a = 1$ was given, and earlier, in [7] a more restricted exact result with $a = 1$ and $t = 1$ was proved. The cases for simple graphs when $a = 1$, $t = 1$, $s \in \{0, 1\}$ were dealt with in [6]. The first result on these lines was the case $a = 1$, $t = 1$, $s = 0$ for simple graphs; it was considered in 1984 and 1986 by Era [5] and Egawa [4], using methods which were different from ours.

Proof of Theorem 6.

(i) We show that

$$\beta(r,s,a,t) \geq \frac{r}{a}(tr+s+c) + (t-1)r$$

where $a \mid tr+s+c$ and $-1 \leq c \leq a-2$.

Let $d = \frac{r}{a}(tr+s+c) + (t-1)r - 1$. We show that, for this value of d, there do not exist t values of x between $\frac{d+s}{r+a}$ and $\frac{d}{r}$. Then, by Theorem 3, it follows that there exist $(d, d+s)$-bipartite multigraphs which do not have $(r, r+a)$-factorizations with x $(r, r+a)$-factors for t different values of x.

We have

$$\frac{d}{r} = \frac{1}{a}(tr+s+c) + (t-1) - \frac{1}{r}$$

and

$$d+s = (r+a)\frac{1}{a}(tr+s+c) - c - r - 1$$

so that

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) - \frac{r+c+1}{r+a}.$$

Since $c+1 < a$ it follows that the values of x which satisfy $\frac{d+s}{r+a} \le x \le \frac{d}{r}$ are $\frac{1}{a}(tr+s+c)+j$ for $0 \le j \le t-2$, so there are indeed fewer than t such values.

(ii) Next we show that $\beta(r,s,a,t) \le \frac{r}{a}(tr+s+c)+(t-1)r$.

Let $d = \frac{r}{a}(tr+s+c)+(t-1)r+k$, where $k \ge 0$. We show that, in this case, there do exist t values of x between $\frac{d+s}{r+a}$ and $\frac{d}{r}$. Then it follows from Theorem 3 that every $(d,d+s)$-bipartite multigraph has an $(r,r+a)$-factorization into x $(r,r+a)$-factors for at least t values of x.

First note that

$$\frac{d}{r} = \frac{1}{a}(tr+s+c)+t-1+\frac{k}{r}$$

and that

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) - \frac{r+c}{r+a} + \frac{k}{r+a}.$$

Therefore if $r+c \ge k \ge 0$ then, since $r+a > r+a-2 \ge r+c$, the values of x lying between $\frac{d+s}{r+a}$ and $\frac{d}{r}$ include

$$\frac{1}{a}(tr+s+c), \dots, \frac{1}{a}(tr+s+c)+t-1$$

so there are at least t values of x. We also note that

$$\frac{d}{r} - \frac{d+s}{r+a} = t-1+\frac{r+c}{r+a}+\frac{ak}{r(r+a)}.$$

Therefore if $\frac{r+c}{r+a}+\frac{ak}{r(r+a)} \ge 1$, i.e. $k \ge (1-\frac{c}{a})r$, then $\frac{d}{r} - \frac{d+s}{r+a} \ge t$.

Since c is an integer, if $c \ne -1$ then all values of $k \ge 0$ satisfy one of the inequalities $k \ge (1-\frac{c}{a})r$ and $r+c \ge k \ge 0$, so it follows from Theorem 3 that $\beta(r,s,a,t) \le \frac{r}{a}(tr+s+c)+t-1$.

Now consider further the case when $c = -1$. If $0 \le k \le r-1$ then, as we just showed, there are t suitable integral values of x. Now suppose that $2r+a \ge k \ge r$. Then

$$\frac{d}{r} \ge \frac{1}{a}(tr+s+c)+(t-1)+1 = \frac{1}{a}(tr+s+c)+t,$$

while

$$\begin{aligned} \frac{d+s}{r+a} &= \frac{1}{a}(tr+s+c)+1-\frac{2r+a-k+c}{r+a} \\ &\le \frac{1}{a}(tr+s+c)+1, \end{aligned}$$

since $c = -1 < 2r + a - k$. So in this case also there are t suitable integral values of x.

The set of inequalities $0 \le k \le r-1$ ($r \le k < 2r+a+c$ when $c = -1$) and $k \ge (1+\frac{1}{a})r$ cover all values of $k \ge 0$. Therefore it follows that

$$\beta(r,s,a,t) \le \frac{r}{a}(tr+s+c)+t-1$$

in this case also.

It now follows that $\beta(r,s,a,t) = \frac{r}{a}(tr+s+c)+t-1$. $\square$

3 Factorizations of pseudographs

In this section we give analogues for certain kinds of pseudographs of Theorem 3 and similar theorems for multigraphs and simple graphs in [6], [7] and [8]. The analogue of Theorem 3 is the following Theorem 7 about $(2r, 2r+2a)$-factorizations of $(2d, 2d+2s)$-pseudographs.

Theorem 7. *Let d, r and s be positive integers, and let s be a non-negative integer.*

(i) If

$$\frac{d+s}{r+a} \le x \le \frac{d}{r},$$

then every $(2d, 2d+2s)$-pseudograph has a $(2r, 2r+2a)$-factorization into x $(2r, 2r+2a)$-factors.

(ii) If

$$x \in \left[\frac{d}{r+a}, \frac{d+s}{r+a}\right) \cup \left(\frac{d}{r}, \frac{d+s}{r}\right]$$

then some $(2d, 2d+2s)$-pseudographs do and some do not have a $(2r, 2r+2a)$-factorization into x $(2r, 2r+2a)$-factors.

(iii) If

$$x \notin \left[\frac{d}{r+a}, \frac{d+s}{r}\right]$$

then no $(2d, 2d+2s)$-pseudograph has a $(2r, 2r+2a)$-factorization into x $(2r, 2r+2a)$-factors.

It would be interesting to know to what extent Theorem 7 remains true if $2r$ is replaced with $2r+1$ or $2a$ is replaced by $2a+1$; in particular, is Theorem 7 still true if $2s$ is replaced by $2s+1$?

It is convenient to prove Theorem 7 by deducing it from Theorem 3 using the following well-known connection between pseudographs and bipartite multigraphs.

Let G be a pseudograph. Pair off the vertices of G of odd degree, and, for each such pair $\{x,y\}$, introduce an extra edge xy. Call the pseudograph obtained this way G^*. Then each component of G^* is Eulerian. Choose an Eulerian circuit of each component of G^* and orient the edges in one direction round each such Eulerian circuit. If $V = V(G^*) = \{v_1, v_2, \ldots, v_r\}$ then construct a bipartite multigraph $B(G^*)$ with vertex sets $U = \{u_1, \ldots, u_r\}$ and $W = \{w_1, \ldots, w_r\}$. If (v_x, v_y) is an oriented edge of G^* then join u_x to w_y in $B(G^*)$ by an edge. If G^* has a loop on v_x, then join u_x to w_x in $B(G^*)$. Now from $B(G^*)$ construct a bipartite multigraph $B(G)$ by deleting each edge of $B(G^*)$ that corresponds to one of the extra edges introduced above in forming G^* from G. Clearly, given a pseudograph G, the extra edges, the Eulerian circuits of the components, and the orientations can all usually be chosen in many different ways, so there are many possibilities for $B(G)$. They all have the property that $|d_{B(G)}(u_i) - d_{B(G)}(w_i)| \leq 1$ for each i, $1 \leq i \leq r$.

On the other hand, given a bipartite multigraph B with vertex sets $U = \{u_1, \ldots, u_r\}$ and $W = \{w_1, \ldots, w_r\}$ satisfying the inequality $|d_B(u_i) - d_B(w_i)| \leq 1$, then it is possible to obtain a pseudograph $G(B)$. Given a pseudograph G, although there are many different possibilities for $B(G)$, reversing the construction will always produce the original pseudograph G again. Thus $G(B(G)) = G$.

We now develop this connection in a more specific way for $(2r, 2r+2a)$-factorizations.

Theorem 8. *A pseudograph G has a $(2r, 2r+2a)$-factorization into x $(2r, 2r+2a)$-factors if and only if a corresponding bipartite multigraph $B(G)$ has an $(r, r+a)$-factorization into x $(r, r+a)$-factors.*

Proof. (i) Suppose G has a $(2r, 2r+2a)$-factorization into x $(2r, 2r+2a)$-factors $F_1, \ldots, F_x$. For $1 \leq i \leq x$, construct a bipartite multigraph $B(F_i)$ corresponding to the factor F_i. Then $B(F_i)$ is an $(r, r+a)$-bipartite multigraph and $(B(F_1), \ldots, B(F_x))$ is an $(r, r+a)$-factorization of a bipartite multigraph $B(G)$.

(ii) Suppose a bipartite multigraph B has an $(r, r+a)$-factorization into x $(r, r+a)$-factors, say $F_1, \ldots, F_x$. For each i, $1 \leq i \leq x$, F_i corresponds to a $(2r, 2r+2a)$-pseudograph $G(F_i)$, and $(G(F_1), \ldots, G(F_x))$ is a $(2r, 2r+2a)$-factorization of $G(B)$. $\square$

We now turn to the proof of Theorem 7.

Proof of Theorem 7.

(i) Let G be a $(2d, 2d+2s)$-pseudograph and let $\dfrac{d+s}{r+a} \leq x \leq \dfrac{d}{r}$. From G we may form a bipartite $(d, d+s)$-multigraph $B(G)$. By Theorem 3(i) $B(G)$ has an $(r, r+a)$-factorization into x $(r, r+a)$-factors. By Theorem 8, this corresponds to a $(2r, 2r+2a)$-factorization of G into x $(2r, 2r+2a)$-factors.

(ii) Let $x \in \left[\dfrac{d}{r+a}, \dfrac{d+s}{r+a}\right) \cup \left(\dfrac{d}{r}, \dfrac{d+s}{r}\right]$. By Theorem 3(ii), some $(d, d+s)$-bipartite multigraphs do and some do not have an $(r, r+a)$-factorization into x $(r,r+a)$-factors. Let B_1 and B_2 be $(d,d+s)$-bipartite multigraphs which do, and do not, respectively, have an $(r,r+a)$-factorization into x $(r,r+a)$-factors. Then, by Theorem 8, $G(B_1)$ and $G(B_2)$ are $(2r,2r+2a)$-pseudographs which do, and do not, respectively, have a $(2r,2r+2a)$-factorization into x $(2r,2r+2a)$-factors.

(iii) Let $x \notin \left[\frac{d}{r+a}, \frac{d+s}{r}\right]$. By Theorem 3(iii), no $(d,d+s)$-bipartite multigraph has an $(r,r+a)$-factorization into x $(r,r+a)$-factors. Therefore, by Theorem 8, no $(2d,2d+2s)$-pseudograph has a $(2r,2r+2a)$-factorization into x $(2r,2r+2a)$-factors. □

We note the following corollaries to Theorem 7.

Corollary 9. *Let d,r,x be positive integers and let s be a non-negative integer. Then every $(2d,2d+2s)$-pseudograph has a $(2r,2r+2a)$-factorization into x $(2r,2r+2a)$-factors if and only if*

$$x \in \left[\frac{d+s}{r+a}, \frac{d}{r}\right].$$

Corollary 10. *Let d,r,x be positive integers and let s be a non-negative integer. Then there is some $(2d,2d+2s)$-pseudograph which has a $(2r,2r+2a)$-factorization into x $(2r,2r+2a)$-factors if and only if*

$$x \in \left[\frac{d}{r+a}, \frac{d+s}{r}\right].$$

We now turn to the analogue of Theorem 6. For positive integers r,a,t and non-negative integer s, let $\psi(r,s,a,t)$ be the smallest integer such that, for each integer $d \geq \psi(r,s,a,t)$, each $(d,d+s)$-pseudograph has an $(r,r+a)$-factorization with x $(r,r+a)$-factors for at least t different values of x. For values of r,s,a,t for which $\psi(r,s,a,t)$ takes no (finite) value, we put $\psi(r,s,a,t) = \infty$.

Theorem 11. *Let r,a,t be positive integers and s a non-negative integer. Let r, s and a all be even. Let c be an even integer such that $a \mid tr+s+c$ and $0 \leq \frac{c}{2} \leq \frac{a}{2} - 1$. Then*

$$\psi(r,s,a,t) = \frac{r}{a}(tr+s+c) + (t-1)r.$$

Remark. Please notice that Theorem 11 and our whole account up to and including Theorem 19 does not use anything peculiar to pseudographs. It could equally well apply to multigraphs without loops, or to simple graphs. We shall make use of this fact in Section 5 about multigraphs and simple graphs.

When $s \in \{0,1\}$ then, as is explained in [6], $\psi(r,s,1,1) = \infty$. Some analogous numbers in the case $t=1$ for multigraphs (where loops are disallowed) were studied by Akiyama and Kano [2], Kano [10] and Cai [3], and good results were obtained. In [6] better bounds for multigraphs, although mostly not best possible, in the case $a = t = 1$, $s \in \{0,1\}$ were found. In [10] Kano showed that a multigraph

G is $(2r, 2r+2a)$-factorizable if and only if G is a $(2rm, 2rm+2am)$-multigraph for some positive integer m. (This follows from a similar theorem of de Werra (see [11]) which says that a bipartite multigraph G is $(r, r+a)$-factorizable if and only if G is an $(rm, rm+am)$-bipartite multigraph for some positive integer m, by using the connection sketched out above between bipartite multigraphs and pseudographs; of course, although not stated as such, Kano's theorem holds for pseudographs, not just for multigraphs.)

In order to prove Theorem 11 more easily, we introduce two further functions, $\psi_e(r,s,a,t)$ and $\gamma(r,s,a,t)$. For integers $t \geq 1$, $r \geq 2$, $a \geq 2$, $s \geq 0$ and r, a, s all even, we let $\psi_e(r,s,a,t)$ be the least even integer such that, for each even integer $d \geq \psi_e(r,s,a,t)$, each $(d, d+s)$-pseudograph has an $(r, r+a)$-factorization with x $(r, r+a)$-factors for at least t different values of x.

For integers $r, a, t \geq 1$ and $s \geq 0$, we let $\gamma(r,s,a,t)$ be the smallest integer such that, for each integer $d \geq \gamma(r,s,a,t)$, each $(2d, 2d+2s)$-pseudograph has a $(2r, 2r+2a)$-factorization into x $(2r, 2r+2a)$-factors for at least t different values of x.

We first determine the value of $\gamma(r,s,a,t)$.

Lemma 12. *Let r, s, a, t be integers with r, a and t positive and s non-negative. Then*

$$\gamma(r,s,a,t) = \frac{r}{a}(tr+s+c) + (t-1)r,$$

where c is such that $a \mid tr+s+c$ and $-1 \leq c \leq a-2$.

Proof. It follows from Theorem 8 that a $(2d, 2d+2s)$-pseudograph G has a $(2r, 2r+2a)$-factorization into x $(2r, 2r+2a)$-factors if and only if a corresponding $(d, d+s)$-bipartite multigraph $B(G)$ has an $(r, r+a)$-factorization into x $(r, r+a)$-factors. Therefore $\gamma(r,s,a,t) = \beta(r,s,a,t)$. But, by Theorem 6, $\beta(r,s,a,t) = \frac{r}{a}(tr+s+c) + (t-1)r$, where $a \mid tr+s+c$ and $-1 \leq c \leq a-2$. □

From Lemma 12 we deduce immediately the following Lemma 13. Lemma 13 is essentially Lemma 12 rephrased.

Lemma 13. *Let r, s, a, t be integers with r, a, t positive and s non-negative. Let r, s, and a all be even. Then*

$$\psi_e(r,s,a,t) = \frac{r}{a}(tr+s+c) + (t-1)r,$$

where c is such that $a \mid tr+s+c$ and $-1 \leq \frac{c}{2} \leq \frac{a}{2} - 2$.

Proof. From the definitions of $\gamma(r,s,a,t)$ and $\psi_e(r,s,a,t)$ it follows that, if r, s, a are all even, then

$$\psi_e(r,s,a,t) = 2\gamma(\frac{r}{2}, \frac{s}{2}, \frac{a}{2}, t),$$

so by Lemma 12,

$$\psi_e(r,s,a,t) = 2\frac{(r/2)}{(a/2)}\left(t\frac{r}{2} + \frac{s}{2} + \frac{c}{2}\right) + (t-1)\frac{r}{2}$$

where c is such that $(a/2) \mid t(r/2)+(s/2)+(c/2)$ (so that c is also even) and $-1 \le \frac{c}{2} \le \frac{a}{2}-2$. Therefore

$$\psi_e(r,s,a,t) = \frac{r}{a}(tr+s+c)+(t-1)r,$$

where c is such that $a \mid tr+s+c$ (so that c is even) and $-1 \le \frac{c}{2} \le \frac{a}{2}-2$. □

Lemma 14. *Let r,s,a,t be integers with r,a,t all positive and s non-negative. Let r, s, and a all be even. Then*

$$\psi_e(r,s+2,a,t) = \begin{cases} \psi_e(r,s,a,t) & \text{if } a \mid rt+s+c,\ 0 \le \frac{c}{2} \le \frac{a}{2}-2, \\ \psi_e(r,s,a,t)+r & \text{if } a \mid rt+s+c,\ \frac{c}{2} = -1. \end{cases}$$

Proof. By Lemma 13

$$\psi_e(r,s+2,a,t) = \frac{r}{a}(tr+(s+2)+c')+(t-1)r$$

where $a \mid tr+(s+2)+c'$ and $-1 \le \frac{c'}{2} \le \frac{a}{2}-2$. Put $c^* = c'+2$. Then

$$\begin{aligned} \psi_e(r,s+2,a,t) &= \frac{r}{a}(tr+(s+2)+(c^*-2))+(t-1)r \\ &= \frac{r}{a}(tr+s+c^*)+(t-1)r \end{aligned}$$

where $a \mid tr+s+c^*$ and $0 \le \frac{c^*}{2} \le \frac{a}{2}-1$. If $0 \le \frac{c^*}{2} \le \frac{a}{2}-2$, then it follows from Lemma 13 that

$$\psi_e(r,s+2,a,t) = \psi_e(r,s,a,t).$$

If $\frac{c^*}{2} = \frac{a}{2}-1$, then put $c^+ = c^*-a$. Then

$$\begin{aligned} \psi_e(r,s+2,a,t) &= \frac{r}{a}(tr+s+c^++a)+(t-1)r \\ &= \frac{r}{a}(tr+s+c^+)+(t-1)r+r \end{aligned}$$

where $a \mid tr+s+c^+$ and $\frac{c^+}{2} = -1$. Therefore, by Lemma 13, in this case we have

$$\psi_e(r,s+2,a,t) = \psi_e(r,s,a,t)+r.$$

□

By definition, when r,s,a are all even, if d is EVEN and $d \ge \psi_e(r,s,a,t)$ then each $(d,d+s)$-pseudograph has an $(r,r+a)$-factorization into x $(r,r+a)$-factors for t different values of x, but $\psi(r,s,a,t)$ has the EXTRA property that if d is ODD and $d \ge \psi(r,s,a,t)$ then each $(d,d+s)$-pseudograph has an $(r,r+a)$-factorization into x $(r,r+a)$-factors for t different values of x. Thus it is clear that $\psi(r,s,a,t) \ge \psi_e(r,s,a,t)-1$ when r,s,a are all even. We note that Theorem 11 tells us that, except when $\frac{c}{2} \neq -1$, $\psi(r,s,a,t) = \psi_e(r,s,a,t)$, but when $\frac{c}{2} = -1$ then $\psi(r,s,a,t) = \psi_e(r,s,a,t) + r$.

Proof of Theorem 11. If $d \geq \psi_e(r,s+2,a,t)$ and if d is even, then any $(d, d+s+2)$-pseudograph has an $(r,r+a)$-factorization with x $(r,r+a)$-factors for t different values of x. If $d \geq \psi_e(r,s+2,a,t)$ and d is odd, then $d-1$ is even and any $(d-1,(d-1)+s+2)$-pseudograph is a $(d', d'+s+2)$-pseudograph for some even $d' \geq \psi_e(r,s+2,a,t)$, and so has an $(r,r+a)$-factorization with x $(r,r+a)$-factors for t different values of x. Thus $\psi(r,s,a,t) \leq \psi_e(r,s+2,a,t)$.

Now let $d = \psi_e(r,s+2,a,t) - 1$ and consider a pseudograph $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \emptyset$, G_1 is a regular pseudograph of degree $d+s$, and G_2 is a regular pseudograph of degree d. Any $(r,r+a)$-factorization of G contains an $(r,r+a)$-factorization of G_1 and an $(r,r+a)$-factorization of G_2.

By Lemma 14, $\psi_e(r,s+2,a,t) = \psi_e(r,s,a,t)$ or $\psi_e(r,s,a,t) + r$. Suppose first that $\psi_e(r,s+2,a,t) = \psi_e(r,s,a,t)$. Let $a \mid rt+s+c$ where, in accordance with Lemma 14, $0 \leq \frac{c}{2} \leq \frac{a}{2} - 2$. Consider G_1. Then

$$\frac{d}{r} = \frac{1}{a}(tr+s+c) + (t-1) - \frac{1}{r},$$

so the number of $(r,r+a)$-factors of G_1 (and therefore G) could have is at most $\frac{1}{a}(tr+s+c) + (t-2)$. Now consider G_2. Then

$$\begin{aligned}
\frac{d+s}{r+a} &= \frac{1}{(r+a)}\frac{1}{a}(tr^2+sr+cr) + \frac{(t-1)r}{r+a} - \frac{1}{r+a} + \frac{s}{r+a} \\
&= \frac{1}{(r+a)}\left(\frac{tr(r+a)}{a} + \frac{s(r+a)}{a} + \frac{c(r+a)}{a} - r - 1 - c\right) \\
&= \frac{1}{a}(tr+s+c) - \frac{r+1+c}{r+a}.
\end{aligned}$$

Since $0 \leq \frac{c}{2} \leq \frac{a}{2} - 2$, it follows that $r+1+c < r+a$ so that $\dfrac{r+1+c}{r+a} < 1$. Therefore the number of $(r,r+a)$-factors in any $(r,r+a)$-factorization is at least $\frac{1}{a}(tr+s+c)$. Therefore the number of different values of x for which G has an $(r,r+a)$-factorization with x $(r,r+a)$-factors is at most $t-1 < t$.

Now suppose that $\psi_e(r,s+2,a,t) = \psi_e(r,s,a,t) + r$. Let $a \mid rt+s+c$ where, again in accordance with Lemma 14, $\frac{c}{2} = -1$. Then

$$\frac{d}{r} = \frac{1}{a}(tr+s+c) + t - \frac{1}{r},$$

so the number of $(r,r+a)$-factors G_1 could have is at most $\frac{1}{a}(tr+s+c) + t - 1$. Now consider G_2. Then

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) - \frac{1+c}{r+a}$$

where $\frac{c}{2} = -1$. Then $1+c = -1$ so $-\frac{1+c}{r+a} > 0$. Therefore the number of $(r, r+a)$-factors G_2 could have is at least $\frac{1}{a}(tr+s+c)+1$. Therefore the number of different values of x for which G has an $(r, r+a)$-factorization with x $(r, r+a)$-factors is at most $t-1 < t$. Thus

$$\psi(r,s,a,t) \geq \psi_e(r,s+2,a,t).$$

Consequently

$$\psi(r,s,a,t) = \psi_e(r,s+2,a,t),$$

so, by Lemma 14,

$$\psi(r,s,a,t) = \begin{cases} \psi_e(r,s,a,t) & \text{if } a \mid rt+s+c,\ 0 \leq \frac{c}{2} \leq \frac{a}{2}-2, \\ \psi_e(r,s,a,t)+r & \text{if } a \mid rt+s+c,\ \frac{c}{2} = -1. \end{cases}$$

Therefore, by Lemma 13,

$$\begin{aligned} \psi(r,s,a,t) &= \begin{cases} \frac{r}{a}(tr+s+c)+(t-1)r & \text{if } a \mid rt+s+c,\ 0 \leq \frac{c}{2} \leq \frac{a}{2}-2, \\ \frac{r}{a}(tr+s+c)+(t-1)r+r & \text{if } a \mid rt+s+c,\ \frac{c}{2} = -1. \end{cases} \\ &= \frac{r}{a}(tr+s+c)+(t-1)r \qquad \text{if } a \mid rt+s+c,\ 0 \leq \frac{c}{2} \leq \frac{a}{2}-1. \end{aligned}$$

□

Corollary 15. *Let r,s,a,t be integers with r,a,t all positive and s non-negative. Let r, s and a be even. Then*

$$\psi(r,s,a,t) = \psi_e(r,s,a,t).$$

We note that Theorem 11 can be re-expressed in the following way.

Theorem 11′. *Let r,s,a,t be integers with r,a,t positive and s non-negative. Let r,s,a be even. Then*

$$\psi(r,s,a,t) = r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r.$$

The remaining task in this section is to remove from Theorem 11 (or 11′) the restriction that s be even. We note the following lemmas.

Lemma 16. *Let r,s,a,t be integers with r,a,t positive and s non-negative. Then*

$$\psi(r,s,a,t) \leq \psi(r,s+1,a,t).$$

Proof. Let $d \geq \psi(r,s+1,a,t)$. Any $(d,d+s)$-pseudograph is also a $(d,d+s+1)$-pseudograph. Thus if all $(d,d+s+1)$-pseudographs have $(r,r+a)$-factorizations with x $(r,r+a)$-factors for at least t values of x, then so do all $(d,d+s)$-pseudographs. Therefore $\psi(r,s+1,a,t) \geq \psi(r,s,a,t)$. □

Lemma 17. *Let r,s,a,t be integers with r,a,t all positive and s non-negative. Let r,a,s be even. If*

$$\left\lceil \frac{rt+s}{a} \right\rceil = \left\lceil \frac{rt+s+2}{a} \right\rceil$$

then $\psi(r,s,a,t) = \psi(r,s+1,a,t) = \psi(r,s+2,a,t)$.

Proof. By Lemma 16, $\psi(r,s,a,t) \leq \psi(r,s+1,a,t) \leq \psi(r,s+2,a,t)$. By Theorem 11 (or 11′), since $\left\lceil \frac{rt+s}{a} \right\rceil = \left\lceil \frac{rt+s+2}{a} \right\rceil$, it follows that $\psi(r,s,a,t) = \psi(r,s+2,a,t)$, so Lemma 17 follows. □

It remains to consider the case when $\lceil \frac{rt+s}{a} \rceil < \lceil \frac{rt+s+2}{a} \rceil$. Since r, s and a are even, this occurs when $a \mid rt+s$. Thus we need to evaluate $\psi(r,s+1,a,t)$ when r and a are even, s is odd and $a \mid rt+s-1$. We do this in Lemma 18.

Lemma 18. *Let r,s,a,t be integers with r,a,t positive and s non-negative. Let r,a be even and s be odd, and let $a \mid rt+s-1$. Then*

$$\psi(r,s,a,t) = r\left(\frac{rt+s-1}{a}\right) + (t-1)r.$$

Proof. Let $d^* = r\left(\frac{rt+s-1}{a}\right) + (t-1)r$. First note that

$$\begin{aligned}
\psi(r,s,a,t) &\geq \psi(r,s-1,a,t) && \text{by Lemma 16,}\\
&= \psi_e(r,s-1,a,t) && \text{by Corollary 15,}\\
&= r\left(\tfrac{rt+s-1}{a}\right) + (t-1)r && \text{by Lemma 13 with } c=0,\\
&= d^*.
\end{aligned}$$

Next notice that, by Lemma 13 (with $c=-2$),

$$\psi_e(r,s+1,a,t) = \frac{r}{a}(tr+(s+1)-2) + (t-1)r,$$

and $a \mid tr+(s+1)-2$.

Thus $\psi_e(r,s+1,a,t) = d^*$. Then, for d even, $d \geq d^*$, any $(d,d+s+1)$-pseudograph has an $(r,r+a)$-factorization into x $(r,r+a)$-factors for t different values of x; therefore any $((d+1),(d+1)+s)$-pseudograph has this property too (since any $((d+1),(d+1)+s)$-pseudograph is a $(d,d+s+1)$-pseudograph), and any $(d,d+s)$-pseudograph has this property (since any $(d,d+s)$-pseudograph is a $(d,d+s+1)$-pseudograph). Therefore, for any integer $d \geq d^*$, any $(d,d+s)$-pseudograph has an $(r,r+a)$-factorization into x $(r,r+a)$-factors for t different values of x. Thus $d^* \geq \psi(r,s,a,t)$, and so

$$\psi(r,s,a,t) = \frac{r}{a}(tr+s-1) + (t-1)r$$

when $a \mid tr+s-1$. □

To sum up our knowledge of $\psi(r,s,a,t)$ when r and a are even, we have:

Theorem 19. *Let r, s, a, t be integers with r, a, t positive and s non-negative. Let r and a be even. Then*

$$\psi(r,s,a,t) = r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r.$$

Proof. This follows from Theorem $11'$, Lemma 17 and Lemma 18. $\square$

4 Bounds for $\psi(r,s,a,t)$ when r, a are not both even

Rather surprisingly, we can find reasonable bounds for $\psi(r,s,a,t)$ when r and a are not both even.

We first note the following lemmas.

Lemma 20. *Let ρ, r, s, a, α, t be integers with ρ, r, a, α, t positive and s non-negative. Let $\rho \leq r \leq r+a \leq \rho+\alpha$. Then*

$$\psi(r,s,a,t) \geq \psi(\rho,s,\alpha,t).$$

Proof. Let $d \geq \psi(r,s,a,t)$. Any $(r, r+a)$-factor of a pseudograph is also a $(\rho, \rho+\alpha)$-factor. Thus if all $(d, d+s)$-pseudographs have an $(r, r+a)$-factorization into x $(r, r+a)$-factors for at least t different values of x, then all $(d, d+s)$-pseudographs have $(\rho, \rho+\alpha)$-factorization into x $(\rho, \rho+\alpha)$-factors for at least t different values of x. Therefore $\psi(r,s,a,t) \geq \psi(\rho,s,\alpha,t)$. $\square$

Two special cases of Lemma 20 are of particular importance.

Lemma 21. *Let r, s, a, t be integers with r, a, t positive and s non-negative. Then*

(i) $\psi(r,s,a,t) \geq \psi(r,s,a+1,t)$.

(ii) $\psi(r,s,a,t) \leq \psi(r+1,s,a-1,t)$.

Proof. (i) corresponds to taking $\rho = r$ and $\alpha = a+1$ in Lemma 20, and then $\rho = r$ and $r+a \leq r+(a+1) = \rho+\alpha$.

(ii) corresponds to taking $\rho = r+1$ and $\alpha = a-1$ in Lemma 20, and then $\rho = r+1$, $r+a = (r+1)+(a-1) = \rho+\alpha$. $\square$

Next we bound $\psi(r,s,a,t)$ when r and a are both odd.

Lemma 22. *Let r, s, a, t be integers with r, t positive, $a \geq 3$ and s non-negative. Let r, a be odd and s be even, let $(r+1)t+s \not\equiv 2 \pmod{a-1}$. Then*

$$\psi(r+1,s,a-1,t) - 1 \leq \psi(r,s,a,t) \leq \psi(r+1,s,a-1,t).$$

Note that, as $r+1$ and $a-1$ are both even, $\psi(r,s,a,t)$ is evaluated in Theorem 19.

Proof. By Lemma 21, $\psi(r,s,a,t) \le \psi(r+1,s,a-1,t)$.

To prove the other inequality, let $d = \psi(r+1,s,a-1,t)-2$, so that d is even. Let F be the $(d,d+s)$-pseudograph with two components, G_1 and G_2, where G_1 has one vertex on which are placed $\frac{d}{2}$ loops, and G_2 has one vertex on which are placed $\frac{d+s}{2}$ loops. Since r and a are both odd and all the edges of G are in fact loops, any $(r,r+a)$-factor of G is actually an $(r+1,r+a)$-factor, i.e. an $((r+1),(r+1)+(a-1))$-factor.

By Lemma 1, it follows that for any $((r+1),(r+1)+(a-1))$-factorization of G into x $((r+1),(r+1)+(a-1))$-factors,

$$\frac{d+s}{(r+1)+(a-1)} \le x \le \frac{d}{r+1}.$$

Since $d = \psi(r+1,s,a-1,t)-2$, it follows from Theorem 19 (since s is even) that

$$d = (r+1)\left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-1)(r+1)-2,$$

so

$$\frac{d}{r+1} = \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-1) - \frac{2}{r+1}.$$

Therefore

$$x \le \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-2).$$

We also have that

$$d+s = (r+1)\left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-1)(r+1)+s-2$$

so that

$$d+s = \frac{(r+1)}{(a-1)}\big(t(r+1)+s+c\big) + (t-1)(r+1)+s-2$$

where $0 \le \frac{c}{2} \le \frac{a-1}{2}-1$ and $a-1 \mid (r+1)t+s+c$.

After some calculation, we find that

$$\frac{d+s}{(r+1)+(a-1)} = \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil - \frac{r+c+3}{r+a}.$$

Since $0 \le \frac{c}{2} \le \frac{a-1}{2}-1$ and $(r+1)t+s \not\equiv 2 \pmod{a-1}$, it follows that $r+c+3 < r+a$, and so

$$x \ge \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil.$$

There are therefore only $t-1$ values that x can take, so there do not exist t values of x for which G has an $((r+1),(r+1)+(a-1))$-factorization into x $((r+1),(r+1)+(a-1))$-factors. Therefore there do not exist t values of x for which G has an $(r,r+a)$-factorization into x $(r,r+a)$-factors. It follows that $d < \psi(r,s,a,t)$.

We now deduce that $\psi(r+1,s,a-1,t)-1<\psi(r,s,a,t)$, so that

$$\psi(r+1,s,a-1,t)-1\leq\psi(r,s,a,t)\leq\psi(r+1,s,a-1,t).$$

$\square$

The missing case of Lemma 22, when $(r+1)t+s\equiv 2 \pmod{a-1}$, is covered less well by Lemma 23:

Lemma 23. *Let r,s,a,t be integers with r,t positive, $a\geq 3$ and $s\geq 2$. Let r,a be odd and $(r+1)t+s\equiv 2 \pmod{a-1}$ (so that s is even). Then*

$$\psi(r+1,s-2,a-1,t)-1\leq\psi(r,s,a,t)\leq\psi(r+1,s-2,a-1,t)+r.$$

Note that $\psi(r+1,s-2,a-1,t)$ can be written down explicitly using Theorem 19.

Proof.

$$\begin{array}{cll}
 & \psi(r+1,s-2,a-1,t)-1 & \\
\leq & \psi(r,s-2,a,t) & \text{by Lemma 22,}\\
\leq & \psi(r,s,a,t) & \text{by Lemma 16,}\\
\leq & \psi(r,s+2,a,t) & \text{by Lemma 16 again,}\\
\leq & \psi(r+1,s+2,a-1,t) & \text{by Lemma 22,}\\
= & \psi(r+1,s-2,a-1,t)+r & \text{by Theorem 19.}
\end{array}$$

$\square$

Theorem 24. *Let r,s,a,t be integers with r,t positive, $a\geq 3$ and s non-negative. Let r,a be odd. Then*

$$\psi(r+1,s,a-1,t)-1\leq\psi(r,s,a,t)\leq\psi(r+1,s,a-1,t)$$

if $(r+1)t+s\not\equiv 1,2 \pmod{a-1}$, and for $i=1$ or 2 and $s\geq i$, then

$$\psi(r+1,s-i,a-1,t)-1\leq\psi(r,s,a,t)\leq\psi(r+1,s-i,a-1,t)+r$$

if $(r+1)t+s\equiv i \pmod{a-1}$.

Note that the bounding terms are given explicitly in each case in Theorem 19.

Proof. If $(r+1)t+s\not\equiv 2 \pmod{a-1}$ and s is even, then the theorem follows from Lemma 22.

If $(r+1)t+s\not\equiv 1 \pmod{a-1}$ and s is odd, then

$$\begin{array}{cll}
 & \psi(r+1,s,a-1,t)-1 & \\
= & \psi(r+1,s-1,a-1,t)-1 & \text{by Theorem 19 since } (r+1)t+s-1\not\equiv 1 \pmod{a-1},\\
\leq & \psi(r,s-1,a,t) & \text{by Lemma 22 since } (r+1)t+(s-1)\not\equiv 2 \pmod{a-1},\\
\leq & \psi(r,s,a,t) & \text{by Lemma 16,}\\
\leq & \psi(r,s+1,a,t) & \text{by Lemma 16 again,}\\
\leq & \psi(r+1,s+1,a-1,t) & \text{by Lemma 22 since } (r+1)t+(s-1)\not\equiv 2 \pmod{a-1},\\
= & \psi(r+1,s,a-1,t) & \text{by Theorem 19 since } (r+1)t+(s+1)-1\not\equiv 1 \pmod{a-1},\\
 & & \text{i.e. } (r+1)t+s\not\equiv 1 \pmod{a-1}.
\end{array}$$

If $(r+1)t+s \equiv 1 \pmod{a-1}$ then s is odd and

$$\begin{array}{rll} & \psi(r+1,s-1,a-1,t)-1 & \\ \leq & \psi(r,s-1,a,t) & \text{by Lemma 22 since } (r+1)t+(s-1) \not\equiv 2 \pmod{a-1}, \\ \leq & \psi(r,s,a,t) & \text{by Lemma 16,} \\ \leq & \psi(r,s+1,a,t) & \text{by Lemma 16 again,} \\ \leq & \psi(r+1,s-1,a-1,r)+r & \text{by Lemma 23 since } (r+1)t+(s+1) \equiv 2 \pmod{a-1}. \end{array}$$

If $(r+1)t+s \equiv 2 \pmod{a-1}$ the theorem follows from Lemma 23. □

Our results and proofs in the remaining cases, when one of r and a is even and the other is odd, are very similar to the case when both r and a are odd, and so we just give brief accounts, accounts which may be filled out by imitating the earlier proofs in obvious ways.

We look next at the case when r is even and a is odd.

Lemma 25. *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and s non-negative. Let r and s be even and a be odd. Let $rt+s \not\equiv 2 \pmod{a-1}$. Then*

$$\psi(r,s,a-1,t)-1 \leq \psi(r,s,a,t) \leq \psi(r,s,a-1,t).$$

Proof. By Lemma 21, $\psi(r,s,a,t) \leq \psi(r,s,a-1,t)$.

To prove the other inequality, let $d = \psi(r,s,a-1,t)-2$, so that d is even. Let G be the $(d,d+s)$-pseudograph with two components, G_1 and G_2, where G_1 has one vertex on which are placed $\frac{d}{2}$ loops, and G_2 has one vertex on which are placed $\frac{d+s}{2}$ loops. Since r is even and a is odd, any $(r,r+a)$-factor of G is actually an $(r,r+(a-1))$-factor.

By Lemma 1, it follows that, for any $(r,r+(a-1))$-factorization of G into x $(r,r+(a-1))$-factors,

$$\frac{d+s}{r+(a-1)} \leq x \leq \frac{d}{r+1}.$$

Using Theorem 19 we find that

$$\frac{d}{r} = \left\lceil \frac{tr+s}{a-1} \right\rceil + t - 1 - \frac{2}{r}$$

so that

$$x \leq \left\lceil \frac{tr+s}{a-1} \right\rceil + t - 2.$$

We also find by arguing as in the proof of Lemma 22 that, for some even integer c such that $0 \leq \frac{c}{2} \leq \frac{a-1}{2} - 1$ and $a-1 \mid rt+s+c$,

$$\frac{d+s}{r+(a-1)} = \left\lceil \frac{tr+s}{a-1} \right\rceil - \frac{c+r+2}{r+(a-1)}.$$

But $c = a-3$ if and only if $rt+s \equiv 2 \pmod{a-1}$ so that, since $rt+s \not\equiv 2 \pmod{a-1}$,

$$x \geq \left\lceil \frac{tr+s}{a-1} \right\rceil,$$

and so there are at most $t-1$ possible values of x.

Therefore there do not exist t values of x for which G has an $(r, r+a)$-factorization into x $(r, r+a)$-factors. Therefore

$$d < \psi(r,s,a,t)$$

and so

$$\psi(r,s,a-1,t) - 1 \leq \psi(r,s,a,t).$$

□

The missing case of Lemma 25, when $rt+s \equiv 2 \pmod{a-1}$ is covered in Lemma 26.

Lemma 26. *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and $s \geq 2$. Let r be even and $rt+s \equiv 2 \pmod{a-1}$ (so that s is even). Then*

$$\psi(r,s-2,a-1,t) - 1 \leq \psi(r,s,a,t) \leq \psi(r,s-2,a-1,t) + r.$$

Proof. Similar to the proof of Lemma 23, but using Lemma 25 instead of Lemma 22. □

Theorem 27. *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and s non-negative. Let r be even and a be odd. Then*

$$\psi(r,s,a-1,t) - 1 \leq \psi(r,s,a,t) \leq \psi(r,s,a-1,t)$$

if $rt+s \not\equiv 1,2 \pmod{a-1}$ and, for $i = 1$ or 2 and $s \geq i$, then

$$\psi(r,s-i,a-1,t) - 1 \leq \psi(r,s,a,t) \leq \psi(r,s-i,a-1,t) + r$$

if $rt+s \equiv i \pmod{a-1}$.

The bounding terms in each case are given explicitly by Theorem 19.

Proof. The proof follows the proof of Theorem 24, using Lemmas 25 and 26 instead of Lemmas 22 and 23. □

Finally we consider the case when r is odd and a is even.

Lemma 28. *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and s non-negative. Let r be odd and a,s be even. Let $(r+1)t+s \not\equiv 2 \pmod{a-2}$. Then*

$$\psi(r+1,s,a-2,t) - 1 \leq \psi(r,s,a,t) \leq \psi(r+1,s,a-2,t).$$

Proof. By Lemma 21,

$$\psi(r,s,a,t) \leq \psi(r+1,s,a-1,t) \leq \psi(r+1,s,a-2,t).$$

To prove the other inequality, let $d = \psi(r+1,s,a-2,t)-2$. Then d is even. Let G be the $(d,d+s)$-pseudograph with two components, G_1 and G_2, where G_1 has one vertex on which are placed $\frac{d}{2}$ loops, and G_2 has one vertex on which are placed $\frac{d+s}{2}$ loops. Since r is even and a is odd, any $(r,r+a)$-factor of G is actually an $((r+1),(r+1)+(a-2))$-factor.

By Lemma 1, it follows that, for any $((r+1),(r+1)+(a-2))$-factorization into x $((r+1),(r+1)+(a-2))$-factors,

$$\frac{d+s}{(r+1)+(a-2)} \leq x \leq \frac{d}{r+1}.$$

Using Theorem 19, it follows that

$$x \leq \left\lceil \frac{t(r+1)+s}{a-2} \right\rceil + t - 2.$$

It also follows (by arguing as in Lemmas 22 and 25) that, for some even integer c such that $0 \leq \frac{c}{2} \leq \frac{a-2}{2} - 1$ and $(a-2) \mid t(r+1)+s+c$,

$$\frac{d+s}{(r+1)+(a-2)} = \left\lceil \frac{t(r+1)+s}{a-2} \right\rceil - \frac{(r+1)+(c+2)}{(r+1)+(a-2)}.$$

But $c = a-4$ if and only if $(r+1)t+s \equiv 2 \pmod{a-2}$ so that, since $(r+1)t+s \not\equiv 2 \pmod{a-2}$,

$$x \geq \left\lceil \frac{t(r+1)+s}{a-2} \right\rceil.$$

Therefore there do not exist t values of x for which G has an $(r,r+a)$-factorization into x $(r,r+a)$-factors. Therefore $d < \psi(r,s,a,t)$ and so $\psi(r+1,s,a-2,t)-1 \leq \psi(r,s,a,t)$. □

The case when $(r+1)t+s \equiv 2 \pmod{a-2}$, missed by Lemma 28, is covered by Lemma 29.

Lemma 29. *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and $s \geq 2$. Let r be odd, a be even, and $(r+1)t+s \equiv 2 \pmod{a-2}$ (so s is even). Then*

$$\psi(r+1,s-2,a-2,t)-1 \leq \psi(r,s,a,t) \leq \psi(r+1,s-2,a-2,t)+r.$$

Proof. This is similar to the proof of Lemma 23, using Lemma 28 instead of Lemma 22. □

Theorem 30. *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and s non-negative. Let r be odd and a be even. Then*

$$\psi(r+1,s,a-2,t)-1 \leq \psi(r,s,a,t) \leq \psi(r+1,s,a-2,t)$$

if $(r+1)t+s \not\equiv 1,2 \pmod{a-2}$, and for $i=1,2$ and $s \geq i$,

$$\psi(r+1,s-i,a-2,t)-1 \leq \psi(r,s,a,t) \leq \psi(r+1,s-i,a-s,t)+r$$

if $(r+1)t+s \equiv i \pmod{a-2}$.

Proof. The proof follows the proof of Theorem 24, using Lemmas 28 and 29 instead of Lemmas 22 and 23. □

5 Multigraphs and simple graphs

In this section we examine the implications of our results on pseudographs have for multigraphs and simple graphs.

First we define analogues of the function $\psi(r,s,a,t)$. For positive integers r,t and non-negative integers a,s, let $\sigma(r,s,a,t)$ be the least integer such that, for each integer $d \geq \sigma(r,s,a,t)$, each $(d,d+s)$-simple graph has an $(r,r+a)$-factorization with x $(r,r+a)$-factors for at least t values of x.

The function $\sigma(r,s,1,t)$ was evaluated in [8], and shown to be given by the formula in:

Theorem 31. *For integers $r,t \geq 1$ and $s \geq 0$,*

$$\sigma(r,s,1,t) = \begin{cases} r(rt+s)+(t-1)r & \text{if } r \text{ is even, } 0 \leq s \leq 1, \\ r(rt+s)+(t-1)r+1 & \text{if } r \text{ is odd, } 0 \leq s \leq 1, \\ r(rt+s)+(t-1)r+r+1 & \text{if } s \geq 2. \end{cases}$$

For positive integers r,t and non-negative integers a,s, let $\mu(r,s,a,t)$ be the least integer such that, for each integer $d \geq \mu(r,s,a,t)$, each $(d,d+s)$-multigraph has an $(r,r+a)$-factorization with x $(r,r+a)$-factors for at least t values of x.

The numbers $\mu(r,0,1,1)$ and $\mu(r,1,1,1)$ were investigated in [6] where bounds were obtained and, for some values of r, the number was determined. The most striking points arising from this are:

(a) if r is odd then $\mu(r,0,1,1) = \sigma(r,0,1,1)$ and, although this is not proven, it seems very likely that $\mu(r,1,1,1) = \sigma(r,1,1,1)$;

(b) if r is even, then, for $s \in \{0,1\}$, $\mu(r,s,1,1)$ is at least approximately $\frac{3}{2}\sigma(r,s,1,1)$.

Kano [10] and Cai [3] also studied $(r,r+a)$-factorizations of $(d,d+s)$-multigraphs; their approach was quite a lot different from ours.

The straightforward relationships between the functions $\sigma(r,s,a,t)$, $\mu(r,s,a,t)$ and $\psi(r,s,a,t)$ are given in the next two theorems.

Theorem 32. *Let r, s, a, t be integers with r, t, a positive and s non-negative. Then*

$$\sigma(r,s,a,t) \leq \mu(r,s,a,t) \leq \psi(r,s,a,t).$$

Proof. This follows from the fact that each simple graph is a multigraph, and each multigraph is a pseudograph. □

Theorem 33. *Let r, a, t be positive integers and s a non-negative integer. Let r and a be even. Then*

$$\sigma(r,s,a,t) = \mu(r,s,a,t) = \psi(r,s,a,t) = r\left\lceil \frac{tr+s-1}{a} \right\rceil + (t-1)r.$$

Proof. We refer back to the remark after the statement of Theorem 11. The whole of the development from Theorem 11 up to Theorem 19 inclusive could apply equally well if the graphs considered were restricted to being multigraphs, or to being simple graphs. Thus the theorem follows from Theorem 19 (and its analogues for simple graphs and multigraphs). □

Theorem 33 enables us to obtain convenient bounds for $\sigma(r,s,a,t)$ and $\mu(r,s,a,t)$ in the case when r and $r+a$ are not both even.

Theorem 34. *Let r, s, a, t be integers with r and a both odd, $r, a \geq 3$, $s \geq 0$, $t \geq 1$. Then*

$$\psi(r-1,s,a+1,t) \leq \sigma(r,s,a,t) \leq \mu(r,s,a,t) \leq \psi(r+1,s,a-1,t).$$

Note that $\psi(r-1,s,a+1,t)$ in Theorem 34 is given explicitly in Theorem 19 (or Theorem 33).

Proof. By Theorem 33,

$$\psi(r-1,s,a+1,t) = \sigma(r-1,s,a+1,t).$$

By the same argument as was used in the proof of Lemma 21, it follows that

$$\sigma(r-1,s,a+1,t) \leq \sigma(r,s,a,t) \leq \sigma(r+1,s,a-1,t).$$

Then, by Theorem 33 again,

$$\sigma(r+1,s,a-1,t) = \psi(r+1,s,a-1,t).$$

Following the same argument for $\mu(r,s,a,t)$ we can obtain:

$$\psi(r-1,s,a+1,t) \leq \mu(r,s,a,t) \leq \psi(r+1,s,a-1,t).$$

Finally we note that, by Theorem 32, $\sigma(r,s,a,t) \leq \mu(r,s,a,t)$. □

Theorem 35. *Let r,s,a,t be integers with r even and a odd. Let $r \geq 1$, $s \geq 0$, $a \geq 3$, $t \geq 1$. Then*

$$\psi(r,s,a+1,t) \leq \sigma(r,s,a,t) \leq \mu(r,s,a,t) \leq \psi(r,s,a-1,t).$$

Proof. This follows similarly since, as in Lemma 21,

$$\sigma(r,s,a+1,t) \leq \sigma(r,s,a,t) \leq \sigma(r,s,a-1,t).$$

□

Theorem 36. *Let r,s,a,t be integers with r odd and a even. Let $r \geq 3$, $s \geq 0$, $a \geq 4$, $t \geq 1$. Then*

$$\psi(r-1,s,a+2,t) \leq \sigma(r,s,a,t) \leq \mu(r,s,a,t) \leq \psi(r+1,s,a-2,t).$$

Proof. The proof is similar:

$$\psi(r-1,s,a+2,t) = \sigma(r-1,s,a+2,t) \leq \sigma(r-1,s,a+1,t) \leq \sigma(r,s,a,t) \leq \cdots$$

$$\cdots \leq \mu(r,s,a,t) \leq \mu(r+1,s,a-1,t) \leq \mu(r+1,s,a-2,t) = \psi(r+1,s,a-2,t).$$

□

Of course, in Theorems 34–36, the upper and lower bounds are given explicitly in Theorem 19 (or Theorem 33).

6 Further comments

Although the bounds for pseudographs we have found are quite good, bounds for multigraphs seem to be harder to obtain, and interest in them seems likely to continue. Multigraph bounds were found by Cai [3] and, as he showed, in some ways these are best possible, but they are not always best possible (see [6] for the case when $a = 1$); they are also expressed in a different way from our results. In Theorem 37 we collect together some bounds for multigraphs which may be readily gleaned from our results. We just give the bounds for $t = 1$, since this is of primary interest, but the bounds when $t > 1$ follow just as easily.

Theorem 37. *Let r,s,a be integers with r,a positive and s non-negative.*

(i) If r and a are even then

$$\mu(r,s,a,1) = r\left\lceil\frac{r+s-1}{a}\right\rceil.$$

(ii) If r and a are odd, $r \geq 3$, $a \geq 3$, then

$$(r-1)\left\lceil\frac{r-1+s}{a+1}\right\rceil \leq \mu(r,s,a,1) \leq (r+1)\left\lceil\frac{r+1+s}{a-1}\right\rceil.$$

(iii) If r is even and a is odd, $a \geq 3$, then

$$r\left\lceil\frac{r+s}{a+1}\right\rceil \leq \mu(r,s,a,1) \leq r\left\lceil\frac{r+s}{a-1}\right\rceil.$$

(iv) If r is odd and a is even, $r \geq 3$, $a \geq 4$, then

$$(r-1)\left\lceil\frac{r-1+s}{a+2}\right\rceil \leq \mu(r,s,a,1) \leq (r+1)\left\lceil\frac{r+1+s}{a-2}\right\rceil.$$

Proof. (i) follows from Theorem 33.

(ii) follows from Theorem 33 and the fact that the analogue of Lemma 21(ii) for $\mu(r,s,a,t)$ is true (it may be established by the same argument).

(iii) follows similarly, using the corresponding analogue to Lemma 21(i) for $\mu(r,s,a,t)$.

(iv) follows similarly, using the analogues of Lemma 21(i) and 21(ii) as follows:

$$\mu(r-1,s,a+2,1) \leq \mu(r-1,s,a+1,1) \leq \mu(r,s,a,1), \text{ and}$$
$$\mu(r,s,a,1) \leq \mu(r+1,s,a-1,1) \leq \mu(r+1,s,a-2,1).$$

□

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